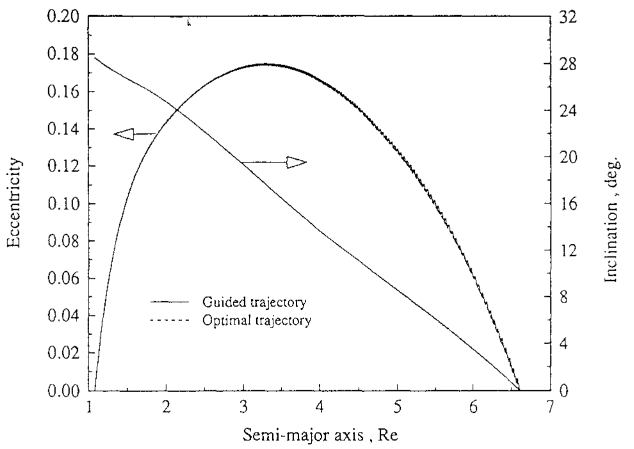

Fig. 1 Eccentricity and inclination vs semimajor axis.

decreases as the semimajor axis (and therefore orbital period) increases. Despite the inherent oscillatory behavior of eccentricity, the guidance law maintains good tracking of the averaged eccentricity profile for the minimum-fuel transfer. The reference inclination profile is tracked almost perfectly without error, as also demonstrated in Fig. 1. The final values of eccentricity and inclination at GEO altitude are 4×10^{-4} and 0.04 deg, respectively. The trip time and final mass are 201.1 days and 5220.6 kg. The optimal trajectory computed by SECKSPOT delivers 5239.5 kg to GEO in 197.8 days. Therefore, the guidance scheme requires 0.4% more fuel and 1.7% more time to complete the transfer. However, SECKSPOT approximates the entire trajectory, utilizing orbital averaging techniques, whereas the inverse dynamics guidance simulation involves accurate numerical integration of the unaveraged state equations.

The resulting in-plane steering amplitude $u_{\max}$ starts small (tangent steering), increases to maintain the positive eccentricity rate, and then becomes negative at the peak eccentricity. The steering amplitude must be negative after the peak eccentricity in order to shift the phase of the sinusoidal steering law and produce a negative eccentricity rate. The out-of-plane steering amplitude $\sigma_{\max}$ steadily increases during the transfer in order to complete the majority of the plane change near the end of the LEO-GEO maneuver.

Summary and Conclusions

A guidance scheme based on inverse dynamics has been devised for a three-dimensional, low-thrust transfer from LEO to GEO. The guidance problem is challenging since the transfer time is extremely long and coasting arcs are present during Earth-shadowing conditions. The guidance strategy utilizes the optimal minimum-fuel trajectory based on the averaged state equations as a reference trajectory. The guidance law uses orbital averaging coupled with a relatively simple steering law to provide accurate closed-loop tracking of the optimal trajectory. The orbit transfer guidance has proven quite effective in simulations performing this difficult maneuver in the context of the oscillatory unaveraged state equations and exhibits near-optimal performance. This guidance scheme could be implemented onboard a solar electric vehicle with modest computational and storage requirements and is therefore a viable candidate for an autonomous, low-thrust orbit transfer guidance system.

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System Design by Linear Exponential Quadratic Gaussian and Loop Transfer Recovery Methodology

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I. Introduction

A ROBUST feedback control system is a system that can satisfy the design requirements within the variation bounds of the actual system dynamics. In the past several years significant methods and advances have been made in integrating time-domain optimization-based control design techniques, such as linear quadratic Gaussian (LQG), with frequency-domain approaches. These integrated frequency-domain and state-space approaches to the design of multi-input, multi-output (MIMO) control system have culminated in the methodology called LQG with loop transfer recovery (LTR). The adoption of the LTR method is to improve the robustness of the LQG regulator with the state observer.[1,2]

In this Note, the linear exponential quadratic Gaussian (LEQG) performance criteria[3–6] and loop transfer recovery (LEQG/LTR) methodology for MIMO robust control system design are developed. It is known[3–6] that, for the LEQG problem, the separation principle can be applied, but the certainty equivalence principle cannot. Furthermore, it was shown[6] that optimal control of the LEQG problem could reduce the maximum and the standard deviation of the miss distance as well as the total rolling angle of a bank-to-turn (BTT) missile. This conclusion was based upon time-domain simulation without any frequency-domain interpretation. Therefore, the first purpose of this Note is to derive the algorithms for the MIMO robust control system design by the LEQG/LTR method, and the second purpose is to study the difference in frequency responses for systems designed by LEQG/LTR and LQG/LTR methodologies.

In this Note, it is shown that the LEQG/LTR method is similar to the LQG/LTR method but with a little modification; e.g., the LEQG/LTR method can also be applied to the design of return ratios at the output or input of the plant. For the former case, the first step is to design the Kalman filter, i.e., manipulating system and measurement noise covariances until a return ratio is obtained that is satisfactory at the plant output. The second step is to synthesize an optimal controller based on LEQG performance criteria. In addition to being able to manipulate the state and control weighting factors, an additional weighting factor can be adjusted to obtain better responses. However, this weighting factor is not included in the LQR/LTR method, causing the certainty equivalence principle to not be held by the LEQG/LTR method. Therefore, one can trade off σ to relax the recovery parameter without lowering the robustness (sometimes the robustness may even be in-

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creased). However, this advantage does not exist with the LQG/LTR method.

An example[7] to design the return ratio at the plant output of a robust flight control system is given for comparison with the results obtained by the LQG/LTR method.

II. General Solution of LEQG Problem

Let the system be represented by the time-invariant state equation

$$\dot{x} = Ax + Bu + \Gamma w \tag{1}$$

and the measurement process

$$y = Cx + v \tag{2}$$

where x is an n-dimensional state vector; u is an n-dimensional control vector; y is a q-dimensional measurement vector; A, B, Γ and C are $n \times n$, $n \times n$, $n \times p$, and $q \times n$ matrices, respectively; w and v are p- and q-dimensional uncorrelated Gaussian white-noise processes with zero mean and covariances

$$E\{w(t)w^T(\tau)\} = W\delta(t-\tau) \tag{3}$$

$$E\{v(t)v^T(\tau)\} = V\delta(t-\tau) \tag{4}$$

and

$$E\{v(t)w^T(\tau)\} = 0 \tag{5}$$

respectively. The problem is to minimize the following LEQG performance criterion[3–6]:

$$J(x, t_0) = \sigma E\left\{\exp\left\{\frac{\sigma}{2}x^T(t_f)S_f x(t_f) + \frac{\sigma}{2}\int_{t_0}^{t_f}[x^T(t)Qx(t) + u^T(t)Ru(t)]\,\mathrm{d}t\right\}\right\} \tag{6}$$

where $\exp\{\cdot\}$ is an exponential function, S_f is $n \times n$ positive semidefinite weighting matrix for the terminal states, Q is $n \times n$ positive semidefinite state weighting matrix, R is an $n \times n$ positive-definite control weighting matrix, and σ is a real number that is also a weighting factor of the LEQG method.

Applying the separation theorem, a Kalman filter can be constructed to produce the optimal estimated state from the noisy measurements, and the state estimation equation can be written as

$$\dot{\hat{x}} = A\hat{x} + Bu + K_f(y - C\hat{x}) \tag{7}$$

Since the original input to the optimal controller x should be replaced by $\hat{x}$, the performance criterion of Eq. (6) is reduced to

$$J(\hat{x}, t_0) = \sigma E\left\{\exp\left\{\frac{\sigma}{2}\hat{x}^T(t_f)S_f\hat{x}(t_f) + \frac{\sigma}{2}\int_{t_0}^{t_f}[\hat{x}^T(t)Q\hat{x}(t) + u^T(t)Ru(t)]\,\mathrm{d}t\right\}\right\} \tag{8}$$

where the original state vector x is replaced by the estimated state vector $\hat{x}$ and K_f is the Kalman filter gain defined as

$$K_f = PC^TV^{-1} \tag{9}$$

where the error variance $P = E\{(x - \hat{x})(x - \hat{x})^T\}$ is propagated forward in time by the Riccati equation

$$\dot{P} = AP + PA^T + \Gamma W \Gamma^T - PC^TV^{-1}CP, \qquad P(t_0) = P_0 \tag{10}$$

Since the Kalman filter is an unbiased estimator, the correction term $y - C\hat{x}$ of the Kalman filter in Eq. (7) may be regarded as an equivalent white noise with zero mean and covariance similar to that of v, i.e.,

$$E\{y - C\hat{x}\} = 0 \tag{11}$$

and

$$E\{(y - C\hat{x})(y - C\hat{x})^T\} = R\delta(t) \tag{12}$$

Let the value function $V(\hat{x}, t)$ be the minimum performance[8] from t to t_f, i.e.,

$$V(\hat{x}, t) = \min_{u(t)} J(\hat{x}, t) \tag{13}$$

Then one can obtain the Hamilton-Jacobi-Bellman equation of the LEQG performance criterion[6] as

$$-V_t(\hat{x}, t) = \min_{u(t)}\left\{\frac{\sigma}{2}[x^T(t)Qx(t) + u^T(t)Ru(t)]V(\hat{x}, t) + V_{\hat{x}}^T(\hat{x}, t)[A\hat{x} + Bu] + \tfrac{1}{2}\operatorname{tr}\left[V_{\hat{x}\hat{x}}(\hat{x}, t)K_f V K_f^T\right]\right\} \tag{14}$$

where the subscript denotes the partial derivation with respect to that variable and tr[·] is a trace operator. Following the optimal control theory,[8] the Hamiltonian function H is related to the derivation of the value function as

$$V_t(\hat{x}, t) = -\min_{u(t)} H(\hat{x}, u, t) \tag{15}$$

Comparing Eqs. (14) and (15), one can find

$$H(\hat{x}, u, t) = \frac{\sigma}{2}[x^T(t)Qx(t) + u^T(t)Ru(t)]V(\hat{x}, t) + V_{\hat{x}}^T(\hat{x}, t)[A\hat{x} + Bu] + \tfrac{1}{2}\operatorname{tr}\left[V_{\hat{x}\hat{x}}(\hat{x}, t)K_f V K_f^T\right] \tag{16}$$

Applying the optimal control theorem, optimal control must satisfy the equation

$$0 = \left.\frac{\partial H}{\partial u}\right|_{u=u^*} \tag{17}$$

So from Eqs. (16) and (17), the optimal control $u^*(t)$ can be obtained as

$$u^*(t) = -\frac{1}{\sigma}R^{-1}B^T V_{\hat{x}}(\hat{x}, t)V^{-1}(\hat{x}, t) \tag{18}$$

Suppose the optimal value function $V(\hat{x}, t)$ is of the form

$$V(\hat{x}, t) = e\sigma\exp\left(\tfrac{1}{2}\sigma\hat{x}^T S\hat{x}\right) \tag{19}$$

where e is a scalar function of time and S is a positive-definite symmetric matrix, both of which can be determined[6] as

$$-\dot{e} = \tfrac{1}{2}e\sigma\operatorname{tr}\left[SK_f V K_f^T\right] \tag{20}$$

$$-\dot{S} = Q + SA + A^TS - S\left(BR^{-1}B^T - \sigma K_f V K_f^T\right)S \tag{21}$$

with the boundary conditions of e and S to be derived from Eqs. (6), (13), and (19), i.e.,

$$e(t_f) = 1 \tag{22}$$

$$S(t_f) = S_f \tag{23}$$

By Eqs. (18) and (19), optimal control can be obtained as a linear combination of the estimated states:

$$u^*(t) = -R^{-1}B^TS\hat{x} \tag{24}$$

where S must satisfy the Riccati equation defined by Eq. (21).

It should be noted that, by Eqs. (9), (10), (21), and (24), if the weighting factor $\sigma \neq 0$ in Eq. (21), then the optimal control gains would take both system and measurement noise covariances (W and V) into consideration, which are different from those obtained by the LQG method, and this is why the certainty equivalence principle cannot be held by the LEQG method, but it provides another degree of freedom for the design to get better performance. Let the value in the parenthesis of Eq. (21) be defined as

$$G = BR^{-1}B^T - \sigma K_f V K_f^T \tag{25}$$

If B and G are nonsingular, then one has an effective control weighting R_{eff} for the LEQG problem defined as

$$R_{\text{eff}} = B^T G^{-1} B \tag{26}$$

From Eq. (25) it can be seen that, if $\sigma < 0$, then the effective control weighting R_{eff} would be decreased.[3] However, by Eqs. (21) and (24) both S and the optimal control gain would be increased. Therefore, the overshoot and the bandwidth of the system would be increased. On the other hand, it is pointed out in this Note that there is a positive upper limit of σ, say $\sigma_{\max}$, such that

$$R_{\text{eff}} > 0 \qquad \text{for} \qquad \sigma < \sigma_{\max} \tag{27}$$

Since if $\sigma \geq \sigma_{\max}$, then G defined in Eq. (25) would become negative semidefinite, the solution of the Riccati equation (21) may not exist, and the guaranteed gain margin (-6 dB $< GM < \infty$ decibels) and phase margin ($|PM| > 60$ deg) of the traditional optimal control system cannot be preserved. Since the certainty equivalence principle cannot be applied by the LEQG/LTR method, for $0 < \sigma < \sigma_{\max}$, by Eq. (26), the effective control weighting factor R_{eff} would be increased, and one can trade off σ to relax (i.e., increase) the recovery parameter ρ_2. Thus by Eq. (21), S will be increased by the factor $\sqrt{\rho_2}$. However, by Eq. (24) the resultant optimal control gain will be decreased by the factor $\sqrt{\rho_2}$. Therefore, the bandwidth and the overshoot of the system can be decreased. Thus, the system is less sensitive to environmental noises.

III. Design Procedures of LEQG/LTR Methodology

From the derivation in Sec. II it can be seen that the LEQG/LTR method is similar to the LQG/LTR method but with a little modification, e.g., the LEQG/LTR method can also be applied for the design of return ratios at the output or input of the plant. For the former case; the first step is to design the Kalman filter, by solving the filter algebraic Riccati equation (FARE) of Eq. (10) as follows:

$$0 = AP + PA^T + \Gamma W \Gamma^T - PC^T V^{-1} CP \tag{28}$$

Then the system and the measurement noise covariances W and V are manipulated, e.g., $W = I$ and $V = \rho_1 I$, until a satisfactory return ratio is obtained at the plant output. The second step is to synthesize an optimal controller based on the LEQG performance criteria, by solving the controller algebraic Riccati equation (CARE) of Eq. (21) as follows:

$$0 = Q + SA + A^T S - S\left(BR^{-1}B^T - \sigma K_f V K_f^T\right)S \tag{29}$$

manipulating the state and the control weighting factors Q and R, e.g., $Q = I$, $R = \rho_2 I$. In addition, another factor, $\sigma K_f V K_f^T$, defined in Eq. (29), can also be designed to obtain better frequency- and/or time-domain responses. If one should design the return ratio at the input of the plant, then one can first set $\sigma = 0$ in Eq. (29) and then design the optimal controller and the Kalman filter just as in the dual method of the LEQG/LTR or LQG/LTR method aforementioned, but one can adjust the σ term in Eq. (29) at the final step. It should be noted that, since the certainty equivalence principle cannot be applied by the proposed method, in addition to adjusting the parameter σ, both the weighting factors ρ_1 and ρ_2 can be modified in the final step to get better performances; i.e., one can trade off σ to relax the recovery parameter ρ_2 (or ρ_1) to lower the optimal control gain (or the Kalman filter gain) to avoid the high-gain problems of the LQG/LTR method.

IV. Example and Simulation Results

In this section, the vertical-plane dynamics of an aircraft[7] is used for the design of autopilot by the proposed LEQG/LTR method. The three inputs are angle of attack, forward acceleration, and deflection of elevation angle. The state variables are altitude, forward speed, pitch angle, pitch rate, and vertical speed. The three outputs are just the first three states to be controlled. The requirements are as follows: 1) closed-loop bandwidth (10 rad/s), 2) integral action in each loop, and 3) well-damped responses for the return ratio at the output of the compensated plant.

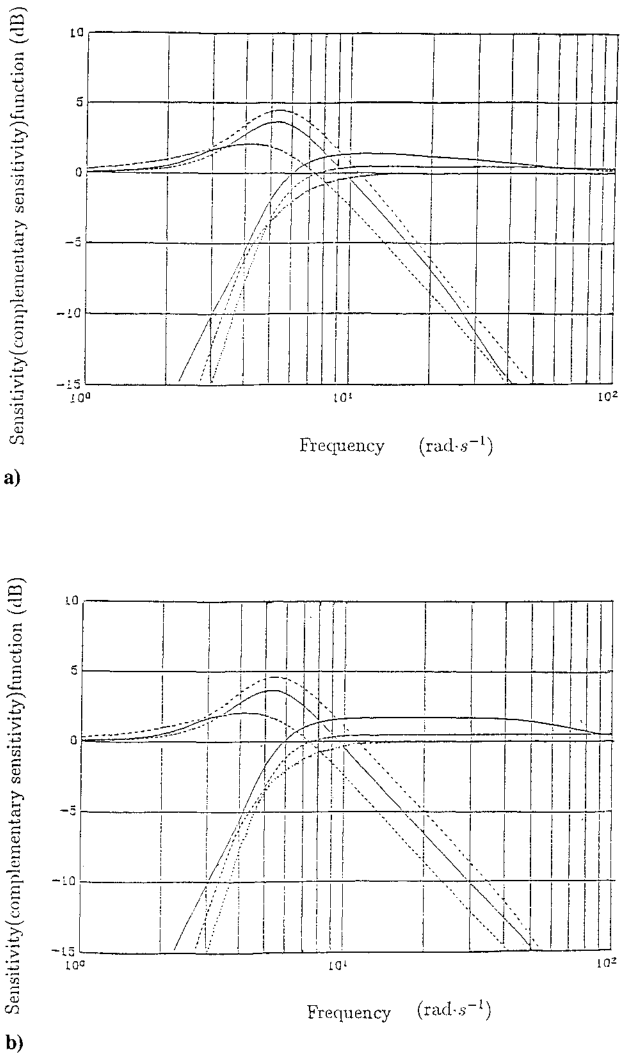

Fig. 1 Principal gains of sensitivity and complementary sensitivity functions obtained with $\rho_2 = 10^{-8}$ by a) LEQG/LTR ($\sigma = 0.1$) and b) LQG/LTR.

Therefore, one should first design the return ratio of a Kalman filter and recover this result at the output of the compensated plant. Since the first step of the LEQG/LTR method is the same as that of the LQG/LTR method, the satisfactory Kalman filter gain K_f ($= K_{f5}$) obtained in Ref. 7 is adopted. Then by the proposed method one can adjust the other two weighting parameters, i.e., ρ_2 and σ. From Figs. 1a and 1b one can see that for some higher control-gain condition,[7] e.g., $\rho_2 = 10^{-8}$ and $\sigma = 0.1$, all LEQG/LTR design results, such as sensitivity and complementary sensitivity functions, are similar to those obtained by the LQG/LTR method. However, if one opts for lower control-gain condition, e.g., $\rho_2 = 10^{-6}$ and $\sigma = 0.1$, from Figs. 2a and 2b, one can see that the results obtained by the proposed method are better:

1) The principal-gain attenuation of the sensitivity function at low frequencies and the complementary sensitivity function at high frequencies can be increased by 1 and 2 dB, respectively.

2) The maximum overshoots of the sensitivity function and the complementary sensitivity function can be reduced by 2 and 1 dB, respectively.

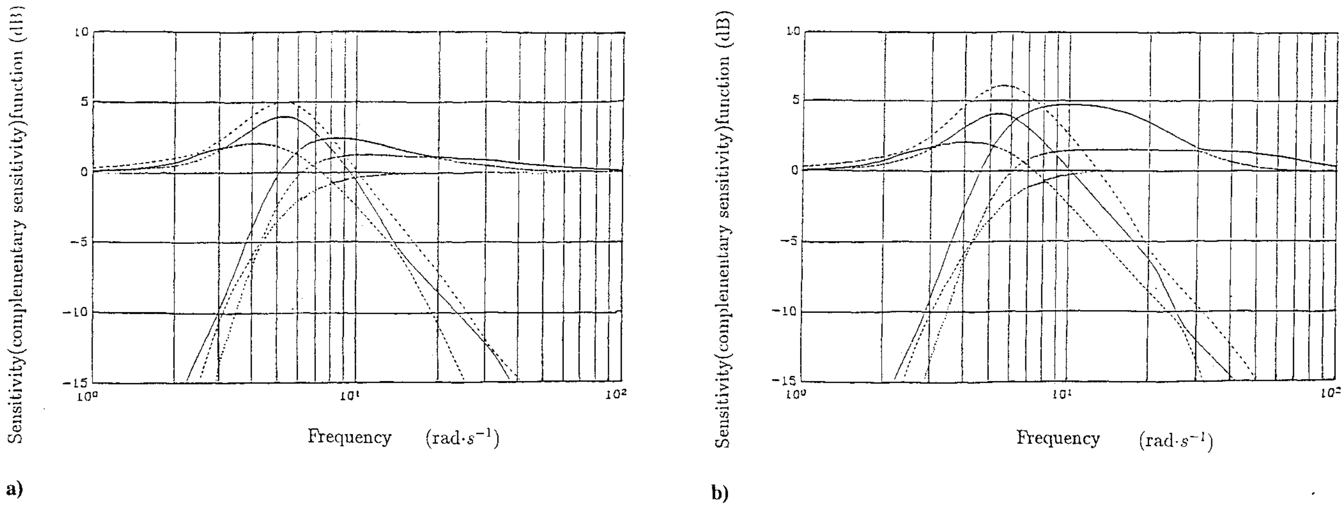

Fig. 2 Principal gains of sensitivity and complementary sensitivity functions obtained with $\rho_2 = 10^{-6}$ by a) LEQG/LTR ($\sigma = 0.1$) and b) LQG/LTR.

3) The separations between the principal-gain curves can be reduced.

V. Discussions and Conclusions

1) It should be noted that, if $\sigma < 0$, then the effective control weighting R_{eff} is reduced, which will produce higher control-gain results. However, σ cannot be too large; otherwise, the effective control weighting R_{eff} would become negative semidefinite, and the guaranteed gain margin and phase margin of the traditional optimal control system cannot be preserved.

2) The major difference between LEQG/LTR and LQG/LTR methodologies is that the certainty equivalence principle cannot be applied by the LEQG/LTR method; thus one can trade off σ to relax the recovery parameter, i.e., to lower the control gain, without lowering the robustness (sometimes the robustness may be even increased, as shown by the previous derivation and the results of simulation). However, this advantage cannot be obtained by the LQG/LTR method.

Acknowledgment

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